Einstein metrics and preserved curvature conditions for the Ricci flow

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1 Introduction

In this note, we study Riemannian manifolds (M,g) with the property that $\mathrm{Ric} = \rho\, g$ for some constant ρ. A Riemannian manifold with this property is called an Einstein manifold. Einstein manifolds arise naturally as critical points of the normalized Einstein-Hilbert action, and have been studied intensively (see e.g. [2]). In particular, it is of interest to classify all Einstein manifolds satisfying a suitable curvature condition. This problem was studied by M. Berger [1]. In 1974, S. Tachibana [9] obtained the following important result:

Theorem 1 (S. Tachibana). *Let (M,g) be a compact Einstein manifold. If (M,g) has positive curvature operator, then (M,g) has constant sectional curvature. Furthermore, if (M,g) has nonnegative curvature operator, then (M,g) is locally symmetric.*

In a recent paper [3], we proved a substantial generalization of Tachibana's theorem. More precisely, it was shown in [3] that the assumption that (M,g) has positive curvature operator can be replaced by the weaker condition that (M,g) has positive isotropic curvature:

Theorem 2. *Let (M,g) be a compact Einstein manifold of dimension $n \geq 4$. If (M,g) has positive isotropic curvature, then (M,g) has constant sectional curvature. Moreover, if (M,g) has nonnegative isotropic curvature, then (M,g) is locally symmetric.*

The proof of Theorem 2 relies on the maximum principle. One of the key ingredients in the proof is the fact that nonnegative isotropic curvature is preserved by the Ricci flow (cf. [5]).

In this note, we show that the first statement in Theorem 2 can be viewed as a special case of a more general principle. To explain this, we fix an integer $n \geq 4$. We

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shall denote by $\mathscr{C}_B(\mathbb{R}^n)$ the space of algebraic curvature tensors on $\mathbb{R}^n$. Furthermore, for each $R \in \mathscr{C}_B(\mathbb{R}^n)$, we define an algebraic curvature tensor $Q(R) \in \mathscr{C}_B(\mathbb{R}^n)$ by

$$Q(R)_{ijkl} = \sum_{p,q=1}^{n} R_{ijpq} R_{klpq} + 2 \sum_{p,q=1}^{n} (R_{ipkq} R_{jplq} - R_{iplq} R_{jpkq}).$$

The term $Q(R)$ arises naturally in the evolution equation of the curvature tensor under the Ricci flow (cf. [6]). The ordinary differential equation $\frac{d}{dt} R = Q(R)$ on $\mathscr{C}_B(\mathbb{R}^n)$ will be referred to as the Hamilton ODE.

We next consider a cone $C \subset \mathscr{C}_B(\mathbb{R}^n)$ with the following properties:

(i) C is closed, convex, and $O(n)$-invariant.

(ii) C is invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R)$.

(iii) If $R \in C \setminus \{0\}$, then the scalar curvature of R is nonnegative and the Ricci tensor of R is non-zero.

(iv) The curvature tensor $I_{ijkl} = \delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}$ lies in the interior of C.

We now state the main result of this note:

Theorem 3. *Let $C \subset \mathscr{C}_B(\mathbb{R}^n)$ be a cone which satisfies the conditions (i)–(iv) above, and let (M,g) be a compact Einstein manifold of dimension n. Moreover, suppose that the curvature tensor of (M,g) lies in the interior of the cone C for all points $p \in M$. Then (M,g) has constant sectional curvature.*

As an example, let us consider the cone

$$C = \{R \in \mathscr{C}_B(\mathbb{R}^n) : R \text{ has nonnegative isotropic curvature}\}.$$

For this choice of C, the conditions (i) and (iv) are trivially satisfied. Moreover, it follows from a result of M. Micallef and M. Wang (see [7], Proposition 2.5) that C satisfies condition (iii) above. Finally, the cone C also satisfies the condition (ii). This was proved independently in [5] and [8]. Therefore, Theorem 2 may be viewed as a subcase of Theorem 3.

2 Proof of Theorem 3

The proof of Theorem 3 is similar to the proof of Theorem 16 in [3]. Let (M,g) be a compact Einstein manifold of dimension n with the property that the curvature tensor of (M,g) lies in the interior of C for all points $p \in M$. If (M,g) is Ricci flat, then the curvature tensor of (M,g) vanishes identically. Hence, it suffices to consider the case that (M,g) has positive Einstein constant. After rescaling the metric if necessary, we may assume that $\mathrm{Ric} = (n-1)\,g$. As in [3], we define an algebraic curvature tensor S by

$$S_{ijkl} = R_{ijkl} - \kappa\,(g_{ik}\,g_{jl} - g_{il}\,g_{jk}), \tag{1}$$

where κ is a positive constant. Let κ be the largest real number with the property that S lies in the cone C for all points $p \in M$. Since the curvature tensor R lies in the interior of the cone C for all points $p \in M$, we conclude that $\kappa > 0$. On the other hand, the curvature tensor S has nonnegative scalar curvature. From this, we deduce that $\kappa \leq 1$.

Proposition 1. *The tensor S satisfies*

$$\Delta S + Q(S) = 2(n-1)\,S + 2(n-1)\,\kappa\,(\kappa-1)\,I,$$

where $I_{ijkl} = g_{ik}\,g_{jl} - g_{il}\,g_{jk}$.

Proof. The curvature tensor of (M,g) satisfies

$$\Delta R + Q(R) = 2(n-1)\,R \tag{2}$$

(see [3], Proposition 3). Using (1), we compute

$$\begin{aligned} Q(S)_{ijkl} &= Q(R)_{ijkl} + 2(n-1)\,\kappa^2\,(g_{ik}\,g_{jl} - g_{il}\,g_{jk}) \\ &\quad - 2\kappa\,(\mathrm{Ric}_{ik}\,g_{jl} - \mathrm{Ric}_{il}\,g_{jk} - \mathrm{Ric}_{jk}\,g_{il} + \mathrm{Ric}_{jl}\,g_{ik}). \end{aligned}$$

Since $\mathrm{Ric} = (n-1)\,g$, it follows that

$$Q(S) = Q(R) + 2(n-1)\,\kappa\,(\kappa-2)\,I. \tag{3}$$

Combining (2) and (3), we obtain

$$\Delta S + Q(S) = 2(n-1)\,R + 2(n-1)\,\kappa\,(\kappa-2)\,I.$$

Since $R = S + \kappa I$, the assertion follows. □

In the following, we denote by $T_S C$ the tangent cone to C at S.

Proposition 2. *At each point $p \in M$, we have $\Delta S \in T_S C$ and $Q(S) \in T_S C$.*

Proof. It follows from the definition of κ that S lies in the cone C for all points $p \in M$. Hence, the maximum principle implies that $\Delta S \in T_S C$. Moreover, since the cone C is invariant under the Hamilton ODE, we have $Q(S) \in T_S C$. □

Proposition 3. *Suppose that $\kappa < 1$. Then S lies in the interior of the cone C for all points $p \in M$.*

Proof. Let us fix a point $p \in M$. By Proposition 2, we have $\Delta S \in T_S C$ and $Q(S) \in T_S C$. Furthermore, we have $-S \in T_S C$ since C is a cone. Putting these facts together, we obtain

$$\Delta S + Q(S) - 2(n-1)\,S \in T_S C.$$

Using Proposition 1, we conclude that

$$2(n-1)\,\kappa\,(\kappa-1)\,I \in T_S C.$$

Since $0 < \kappa < 1$, it follows that $-2I \in T_S C$. On the other hand, I lies in the interior of the tangent cone $T_S C$. Hence, the sum $-2I + I = -I$ lies in the interior of the tangent cone $T_S C$. By Proposition 5.4 in [4], there exists a real number $\varepsilon > 0$ such that $S - \varepsilon I \in C$. Therefore, S lies in the interior of the cone C, as claimed. $\square$

Proposition 4. *The algebraic curvature tensor S defined in (1) vanishes identically.*

Proof. By definition of κ, there exists a point $p_0 \in M$ such that $S \in \partial C$ at p_0. Hence, it follows from Proposition 3 that $\kappa = 1$. Consequently, the Ricci tensor of S vanishes identically. Since $S \in C$ for all points $p \in M$, we conclude that S vanishes identically. $\square$

Since S vanishes identically, the manifold (M, g) has constant sectional curvature. This completes the proof of Theorem 3.

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